

REGULARITY ACTION OF ABELIAN LINEAR GROUPS ON \mathbb{C}^n

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ABSTRACT. In this paper, we give a characterization of the action of any abelian subgroup G of $GL(n, \mathbb{C})$ on \mathbb{C}^n . We prove that any orbit of G is regular with order $m \leq 2n$. Moreover, we give a method to determine this order. In the other hand, we specify the region of all orbits which are isomorphic. If G is finitely generated, this characterization is explicit.

1. Introduction

Let $GL(n, \mathbb{C})$ be the group of all reversible square matrix over \mathbb{C} with order n and let G be an abelian subgroup of $GL(n, \mathbb{C})$. There is a natural linear action $GL(n, \mathbb{C}) \times \mathbb{C}^n : \longrightarrow \mathbb{C}^n$, $(A, v) \longmapsto Av$. For a vector $v \in \mathbb{C}^n$, denote by $G(v) = \{Av, A \in G\} \subset \mathbb{C}^n$ the *orbit* of G through v . A subset $E \subset \mathbb{C}^n$ is called *G-invariant* if $A(E) \subset E$ for any $A \in G$; that is E is a union of orbits. Denote by \overline{E} (resp. $\overset{\circ}{E}$) the closure (resp. interior) of E .

An orbit γ is called *regular* with order m if for every $v \in \gamma$ there exists an open set O containing v such that $\overline{\gamma} \cap O$ is a manifold with dimension m over \mathbb{R} . In particular, γ is locally dense in \mathbb{C}^n if and only if $m = 2n$, and γ is discrete if and only if $m = 0$. Notice that, the closure of a regular orbit is not necessary a manifold (see example 8.4). We say that the action of G is *regular* on \mathbb{C}^n if every orbit of G is regular. Here, the question to investigate is the following:

- (1) *The orbits of G are they regular?*
- (2) *If G has a regular orbit, how can we determine its order?*

The notion of regular orbit is a generalization of non exceptional orbit defined for the action of any group of homeomorphism on a topological space X . A nonempty compact subset $Y \subset X$ is a minimal set if for every $y \in Y$ the orbit of y is dense in Y . In [10], Gottschalk discussed the question of what sets can be minimal sets. A minimal set which is a Cantor set is called an exceptional set. Their dynamics were recently initiated for some classes in different point of view, (see for instance, [3],[4],[5],[6],[7],[9]).

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For a subset $E \subset \mathbb{C}^n$, denote by $\text{vect}(E)$ the vector subspace of \mathbb{C}^n generated by all elements of E . For every $u \in \mathbb{C}^n$ denote by;

- $E(u) = \text{vect}(G(u))$
- $r(u) = \dim(E(u))$ over \mathbb{C} .
- $\text{rank}(G(u)) = \dim(E(u))$.

S.Chihi proved in [12] that for an abelian linear action, every orbit γ is contained in a locally closed sub-manifold $V(\gamma)$ such that $\gamma \subset V(\gamma) \subset \overline{\gamma}$. Moreover, he shows that if $\overline{\gamma_1} = \overline{\gamma_2}$ then γ_1 and γ_2 are isomorphic and $\text{rank}(\gamma_1) = \text{rank}(\gamma_2)$.

The purpose of this paper is to give a complete answer to the above question for any abelian subgroup of $GL(n, \mathbb{C})$. In [1], the authors present a global dynamic of every abelian subgroup of $GL(n, \mathbb{C})$ and in [2], they gave a characterization of existence of dense orbit for any abelian subgroup of $GL(n, \mathbb{C})$. Our main result is viewed as continuation of work in [1] and [2]. We found similar result given in [12] as a consequence of Theorem 1.5, and we prove that every orbit is regular and we characterize its order m . If G is finitely generated, this characterization is explicit.

Denote by:

- $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.
- $e^{(k)} = [e_1^{(k)}, \dots, e_r^{(k)}]^T \in \mathbb{C}^{n+1}$ where

$$e_j^{(k)} = \begin{cases} 0 \in \mathbb{C}^{n_j} & \text{if } j \neq k \\ e_{k,1} & \text{if } j = k \end{cases} \quad \text{for every } 1 \leq j, k \leq r.$$

- $M_n(\mathbb{C})$ the set of all square matrix of order $n \geq 1$ with coefficients in \mathbb{C} .
- $M_{p,q}(\mathbb{C})$ the set of all matrix having p lines and q colons with coefficients in \mathbb{C} .
- $\mathbb{T}_m(\mathbb{C})$ the set of matrices over \mathbb{C} of the form

$$\begin{bmatrix} \mu & & & 0 \\ a_{2,1} & \mu & & \\ \vdots & \ddots & \ddots & \\ a_{m,1} & \dots & a_{m,m-1} & \mu \end{bmatrix} \quad (1)$$

- $\mathbb{T}_m^*(\mathbb{C})$ the group of matrices of the form (1) with $\mu \neq 0$.

Let $r \in \mathbb{N}^*$ and $\eta = (n_1, \dots, n_r) \in (\mathbb{N}^*)^r$ such that $\sum_{i=1}^r n_i = n$. Denote by:

- $\mathcal{K}_{\eta,r}(\mathbb{C}) = \mathbb{T}_{n_1}(\mathbb{C}) \oplus \dots \oplus \mathbb{T}_{n_r}(\mathbb{C})$.
- $\mathcal{K}_{\eta,r}^*(\mathbb{C}) = \mathcal{K}_{\eta,r}(\mathbb{C}) \cap GL(n, \mathbb{C})$.
- $\mathcal{C}_0 = (e_1, \dots, e_n)$ the canonical basis of \mathbb{C}^n .

The author have proved in [2], that for every abelian subgroup of $GL(n, \mathbb{C})$ there exists $P \in GL(n, \mathbb{C})$ such that $P^{-1}GP$ is a subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$ for some $r \in \mathbb{N}^*$ and $\eta = (n_1, \dots, n_r) \in (\mathbb{N}^*)^r$ (see Proposition 2.8). We say

that $\tilde{G} = P^{-1}GP$ is a normal form of G . We let

$$- \mathfrak{g} = \exp^{-1}(G) \cap [P(\mathcal{K}_{\eta,r}(\mathbb{C}))P^{-1}]$$

$$- \mathfrak{g}_u = \{Bu, B \in \mathfrak{g}\}, u \in \mathbb{C}^n.$$

One has $\exp(\mathfrak{g}) = G$ (see Lemma 2.7).

For any closed additive subgroup F of \mathbb{C}^n , we say that $\dim(F) = s$, if s is the bigger dimension of all vector spaces over \mathbb{R} contained in F . Moreover, if F is considered as a manifold over \mathbb{R} , then $\dim(F) = s$. (See Proposition 2.4).

Finally, consider the following rank condition on a collection of vectors $u_1, \dots, u_p \in \mathbb{R}^n$, $p > n$. Suppose that (u_1, \dots, u_n) is a basis of \mathbb{R}^n and there exists $0 \leq m \leq n$ such that $u_k = \sum_{j=1}^m \alpha_{k,j} u_{n-m+j}$, for every $n+1 \leq k \leq p$, $\alpha_{k,j} \in \mathbb{R}^*$.

• We say that $u_1, \dots, u_p \in \mathbb{R}^n$ satisfy *property $\mathcal{D}(m)$* if and only if for every $(t_1, \dots, t_m, s_1, \dots, s_{p-n}) \in \mathbb{Z}^{p-n+m} - \{0\}$:

$$\text{rank} \left(\begin{bmatrix} 1 & 0 & \dots & 0 & \alpha_{n+1,1} & \dots & \dots & \alpha_{p,1} \\ 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & \alpha_{n+1,m} & \dots & \dots & \alpha_{p,m} \\ t_1 & \dots & \dots & t_m & s_1 & \dots & \dots & s_{p-n} \end{bmatrix} \right) = m+1.$$

and this is equivalent by Lemma 5.3 to say that $\mathbb{Z}u_{n-m+1} + \dots + \mathbb{Z}u_p$ is dense in $\mathbb{R}u_{n-m+1} \oplus \dots \oplus \mathbb{R}u_n$.

• For every permutation $\sigma \in \mathcal{S}_p$ we say that $u_{\sigma(1)}, \dots, u_{\sigma(p)} \in \mathbb{R}^n$ satisfy *property $\mathcal{D}(m)$* if $u_1, \dots, u_p \in \mathbb{R}^n$ satisfy *property $\mathcal{D}(m)$* .

For a vector $v \in \mathbb{C}^n$, we write $v = \text{Re}(v) + i\text{Im}(v)$ where $\text{Re}(v), \text{Im}(v) \in \mathbb{R}^n$.

Let $\theta : \mathbb{C}^n \longrightarrow \mathbb{R}^{2n}$ be the isomorphism, defined by

$$\theta(z_1, \dots, z_n) = (\text{Re}(z_1), \dots, \text{Re}(z_n); \text{Im}(z_1), \dots, \text{Im}(z_n)).$$

• We say that $v_1, \dots, v_p \in \mathbb{C}^n$ satisfy *property $\mathcal{D}(m)$* , if $\theta(v_1), \dots, \theta(v_p) \in \mathbb{R}^{2n}$ satisfy *property $\mathcal{D}(m)$* .

Our principal results can be stated as follows:

Theorem 1.1. *Let G be an abelian subgroup of $GL(n, \mathbb{C})$. Then for every $u \in \mathbb{C}^n$, there exists a G -invariant dense open subset U_u of $E(u)$, containing u , such that:*

- (i) *For every $v \in U_u$, we have $E(v) = E(u)$.*
- (ii) *All orbit in U_u are isomorphic.*
- (iii) *$E(u) \setminus U_u$ is a union of at most $r(u)$ G -invariant vector subspaces of $E(u)$ with dimension $r(u) - 1$ over \mathbb{C} .*

By the following Theorem, the order of any orbit is equal to dimension of a closed additive subgroup of \mathbb{C}^n , over \mathbb{R} .

Theorem 1.2. *Let G be an abelian subgroup of $GL(n, \mathbb{C})$ and $u \in \mathbb{C}^n$. The following are equivalent:*

- (i) *$G(u)$ is regular with order m .*
- (ii) *For every $v \in U_u$, $G(v)$ is regular with order m .*
- (iii) *$\dim(\overline{G(u)}) = m$ over \mathbb{R} .*

As a consequence from Theorem 1.2 we prove, by the following corollary, that the action of any abelian subgroup of $GL(n, \mathbb{K})$ on \mathbb{K}^n is regular ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}).

Corollary 1.3. *Let G be an abelian subgroup of $GL(n, \mathbb{C})$ (resp. $GL(n, \mathbb{R})$). Then every orbit of G is regular with order $0 \leq m \leq 2n$ (resp. $0 \leq m \leq n$).*

Where $GL(n, \mathbb{R})$ denotes the group of all reversible square matrix over \mathbb{R} with order n .

Corollary 1.4. *Let G be an abelian subgroup of $GL(n, \mathbb{K})$ ($\mathbb{K} = \mathbb{R}$ or \mathbb{C}).*

- (i) *If $\mathbb{K} = \mathbb{R}$ then an orbit $G(u)$ is regular with order $m = n$ if and only if it is locally dense.*
- (ii) *If $\mathbb{K} = \mathbb{C}$ then an orbit $G(u)$ is regular with order $m = 2n$ if and only if it is dense in \mathbb{C}^n .*

Theorem 1.5. *Let G be an abelian subgroup of $GL(n, \mathbb{C})$ and $u \in \mathbb{C}^n$. Then the closure $\overline{G(u)}$ is a vector subspace of \mathbb{C}^n if and only if $G(u)$ is regular with order $2r(u)$.*

For a finitely generated subgroup $G \subset GL(n, \mathbb{C})$, such that its normal form $\tilde{G} = P^{-1}GP$ is a subgroup of $\mathcal{K}_{\eta, r}^*(\mathbb{C})$. We give an explicit condition to determine the order of any orbit.

Theorem 1.6. *If G is an abelian subgroup of $GL(n, \mathbb{C})$ generated by A_1, \dots, A_p and let $B_1, \dots, B_p \in \mathfrak{g}$ such that $A_1 = e^{B_1}, \dots, A_p = e^{B_p}$. Then for every $u \in \mathbb{C}^n \setminus \{0\}$. The following are equivalent:*

- (i) *$G(u)$ is regular with order m .*
- (ii) *$B_1 u, \dots, B_p u, 2i\pi P(e^{(1)}), \dots, 2i\pi P(e^{(r)})$ satisfy property $\mathcal{D}(m)$.*

$$(iii) \quad g_u = \sum_{k=1}^p \mathbb{Z}(B_k u) + \sum_{k=1}^r 2i\pi \mathbb{Z}P(e^{(k)}) \text{ and } \dim(\overline{g_u}) = m.$$

This paper is organized as follows: In Section 2, we give preliminary results. In Section 3, we prove the Theorem 1.1. A parametrization of an abelian subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$ is given in section 4. The proof of Theorems 1.2, 1.5 and 1.6 and Corollaries 1.3 and 1.4 are done in Section 5. Section 6 is devoted to give some examples.

2. Preliminary results

We present some results for abelian subgroup of $GL(n, \mathbb{C})$.

Proposition 2.1. ([8], Proposition 7') *Let $A \in M_n(\mathbb{C})$. Then if no two eigenvalues of A have a difference of the form $2ik\pi$, $k \in \mathbb{Z} \setminus \{0\}$, then $\exp : M_n(\mathbb{C}) \rightarrow GL(n, \mathbb{C})$ is a local diffeomorphism at A .*

Corollary 2.2. *The restriction $\exp|_{\mathcal{K}_{\eta,r}(\mathbb{C})} : \mathcal{K}_{\eta,r}(\mathbb{C}) \rightarrow \mathcal{K}_{\eta,r}^*(\mathbb{C})$ is a local diffeomorphism.*

Proof. The proof results from Proposition 2.1 and the fact that $\exp|_{\mathcal{K}_{\eta,r}(\mathbb{C})} = \exp|_{\mathbb{T}_{n_1}(\mathbb{C})} \oplus \cdots \oplus \exp|_{\mathbb{T}_{n_r}(\mathbb{C})}$. \square

Proposition 2.3. ([11], Theorem 2.1) *Let H be a discrete additive subgroup of \mathbb{C}^n . Then there exist a basis (u_1, \dots, u_n) of \mathbb{C}^n and $1 \leq r \leq n$ such that $H = \sum_{k=1}^r \mathbb{Z}u_k$.*

Proposition 2.4. ([11], Theorem 3.1) *Let F be a closed additive subgroup of \mathbb{C}^n . Then there exist a vector subspace V of \mathbb{C}^n , over \mathbb{R} contained in F and a vector subspace W of \mathbb{C}^n , over \mathbb{R} , such that:*

- (i) $W \oplus V = \mathbb{C}^n$.
- (ii) $F \cap W$ is a discrete subgroup of \mathbb{C}^n and $F = (F \cap W) \oplus V$.

For any closed additive subgroup F of \mathbb{C}^n , we have $\dim(F) = \dim(V)$.

Corollary 2.5. (Under above notations) *For every $y \in F \cap W$, there exist an open subset O_y of \mathbb{C}^n such that $O_y \cap F = \{y\} + V$. In particular $O_0 \cap F = V$.*

Proof. We have $F = (F \cap W) \oplus V$, where $F \cap W$ is a discrete subgroup of W . By Proposition 2.3 there exists a basis (u_1, \dots, u_p) of W and $1 \leq s \leq p$ such that $F \cap W = \mathbb{Z}u_1 \oplus \cdots \oplus \mathbb{Z}u_s$. Let $y = m_1 u_1 + \dots + m_s u_s \in F \cap W$, take

$$O_y = \left] m_1 - \frac{1}{2}, m_1 + \frac{1}{2} \right[u_1 \oplus \cdots \oplus \left] m_p - \frac{1}{2}, \frac{1}{2} + m_p \right[u_p \oplus V.$$

It follows that O_y is an open subset of \mathbb{C}^n such that $O_y \cap F = V + \{y\}$. The proof is complete. \square

Proposition 2.6. *For every $u \in \mathbb{C}^n$, there exist two vector subspaces V and W of \mathbb{C}^n over \mathbb{R} , such that V is contained in $\overline{g_u}$ and $W \oplus V = \mathbb{C}^n$, satisfying:*

- i) $\overline{g_u} = (\overline{g_u} \cap W) \oplus V$, with $\overline{g_u} \cap W$ is discrete.
- ii) For every $\lambda \in \mathbb{R}$, we have $\overline{g_{\lambda u}} = (\overline{g_{\lambda u}} \cap W) \oplus V$, with $\overline{g_{\lambda u}} \cap W$ is discrete subgroup of W .

The proof uses the following lemma.

Lemma 2.7. ([2], Lemmas 4.1 and 4.2)

- (i) If G is an abelian subgroup of $GL(n, \mathbb{C})$ then for every $u \in \mathbb{C}^n$, g_u is an additive subgroup of \mathbb{C}^n .
- (ii) $\exp(g) = G$.

Proof of Proposition 2.6. The proof of (i) results from Lemma 2.7.(i) and Proposition 2.4.

The proof of (ii) follows from the fact that $g_{\lambda u} = \lambda g_u$, $\lambda W = W$ and $\lambda V = V$, for every $\lambda \in \mathbb{R}$. \square

Let the fundamental result proved in [2]:

Proposition 2.8. ([2], Proposition 2.3) *Let G be an abelian subgroup of $GL(n, \mathbb{C})$. Then there exists $P \in GL(n, \mathbb{C})$ such that $P^{-1}GP$ is an abelian subgroup of $\mathcal{K}_{\eta, r}^*(\mathbb{C})$, for some $r \in \{1, \dots, n\}$ and $\eta \in (\mathbb{N}^*)^r$.*

3. Proof of Theorem 1.1

Throughout this section, suppose that G is an abelian subgroup of $\mathcal{K}_{\eta, r}^*(\mathbb{C})$ for some $r \in \mathbb{N}^*$ and $\eta = (n_1, \dots, n_r) \in (\mathbb{N}^*)^r$.

Every $A \in G$ has the form $A = \text{diag}(A_1, \dots, A_r)$ with $A_k \in \mathbb{T}_{n_k}(\mathbb{C})$ $k = 1, \dots, r$. Denote by:

- $G_k = \{A_k, A \in G\}$, $k = 1, \dots, r$.
- $E(u_k) = \text{vect}(G_k(u_k))$, for every $u = [u_1, \dots, u_r]^T \in \mathbb{C}^n$.
- $\mathcal{G} = \text{vect}(G)$, the vector subspace of $\mathcal{K}_{\eta, r}(\mathbb{C})$ generated by all elements of G .
- $U = \prod_{k=1}^r \mathbb{C}^* \times \mathbb{C}^{n_k-1}$.

Lemma 3.1. *Let G be an abelian subgroup of $\mathcal{K}_{\eta, r}^*(\mathbb{C})$. Then for every $u \in \mathbb{C}^n$, $E(u)$ is G -invariant.*

Proof. Suppose that $E(u)$ is generated by A_1u, \dots, A_pu , with $A_k \in G$, $1 \leq k \leq p$. Let $w = \sum_{k=1}^p \alpha_k A_k u \in E(u)$ and $B \in G$, then $Bw = \sum_{k=1}^p \alpha_k B A_k u$. Since $B A_k u \in G(u) \subset E(u)$ then it has the form $B A_k u = \sum_{j=1}^p \beta_{k,j} A_j u$, $\beta_{k,j} \in \mathbb{C}$, so $Bw = \sum_{1 \leq k, j \leq p} \alpha_k \beta_{k,j} A_j u \in E(u)$. \square

Lemma 3.2. ([12], Proposition 3.1) *Let G be an abelian subgroup of $GL(n, \mathbb{C})$, $u \in \mathbb{C}^n$ and $v \in E(u)$.*

- (i) *Then there exist $B \in \mathcal{G}$ such that $Bu = v$.*
- (ii) *If $E(u) = E(v)$, then $G(u)$ and $G(v)$ are isomorphic.*

Proposition 3.3. *Let G be an abelian subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$ and $u \in U$ such that $E(u) = \mathbb{C}^n$. Then for every $v \in U$ there exists $B \in \mathcal{G} \cap GL(n, \mathbb{C})$ such that $Bu = v$. In particular, $E(v) = \mathbb{C}^n$.*

Lemma 3.4. *Let G be an abelian subgroup of $\mathbb{T}_n^*(\mathbb{C})$ and $u \in \mathbb{C}^* \times \mathbb{C}^{n-1}$. Then for every $v \in \mathbb{C}^* \times \mathbb{C}^{n-1}$ there exist $B \in \mathcal{G} \cap GL(n, \mathbb{C})$ such that $Bu = v$.*

Proof. Let $v \in \mathbb{C}^* \times \mathbb{C}^{n-1}$ since $E(u) = \mathbb{C}^n$, by lemma 3.2, there exist $B \in \mathcal{G}$ such that $Bu = v$. Since $\mathbb{T}_n(\mathbb{C})$ is a vector space so $\mathcal{G} \subset \mathbb{T}_n(\mathbb{C})$. Write $u = [x_1, \dots, x_n]^T$, $v = [y_1, \dots, y_n]^T$ and

$$B = \begin{bmatrix} \mu_B & & & 0 \\ a_{2,1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_{n,1} & \dots & a_{n,n-1} & \mu_B \end{bmatrix},$$

then $\mu_B = \frac{y_1}{x_1} \neq 0$, hence $B \in GL(n, \mathbb{C})$. As $\mathcal{G} \subset \mathcal{C}(G)$, $B(E(u)) = E(v)$. \square

Proof of Proposition 3.3. Write $u = [u_1, \dots, u_r]^T$ and let $v = [v_1, \dots, v_r]^T \in U$, with $u_k, v_k \in \mathbb{C}^{n_k}$. Since $E(u) = \mathbb{C}^n$, then $E(u_k) = \mathbb{C}^{n_k}$ for every $k = 1, \dots, r$. As $\mathcal{G} = \mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_r$ and $v_k \in \mathbb{C}^* \times \mathbb{C}^{n_k-1}$, where $\mathcal{G}_k = Vect(G_k)$. Hence the proof results from Lemma 3.4. In particular, $E(v) = B(E(u)) = \mathbb{C}^n$. \square

3.1. Proof of Theorem 1.1. Let $u \in \mathbb{C}^n$. By Proposition 2.8, we can assume that $G \subset \mathcal{K}_{\eta,r}^*(\mathbb{C})$ and $E(u) = \mathbb{C}^n$, otherwise, by Lemma 3.1 we replace G by the restriction $G_{/E(u)}$. By construction, $\mathbb{C}^n \setminus U$ is union of r G -invariant vector subspaces of \mathbb{C}^n with dimension $n-1$, then we proves (iii) and we deduce that $u \in U$. Now, for every $v \in U$, there exists by Proposition 3.3, $B \in \mathcal{G} \cap GL(n, \mathbb{C})$ such that $Bu = v$. Hence $E(v) = B(E(u)) = \mathbb{C}^n$ and $G(v) = B(G(u))$, this proves (i) and (ii). \square

4. Parametrization

Let G be an abelian subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$ and $u \in \mathbb{C}^n$, by Lemma 3.1, $E(u)$ is G -invariant, so consider the linear map

$$\begin{aligned} \Phi_u : \text{vect}(G_{/E(u)}) &\longrightarrow E(u) \\ A &\mapsto Au \end{aligned}$$

Proposition 4.1. *For every $u \in \mathbb{C}^n \setminus \{0\}$, Φ_u is a linear isomorphism.*

Proof. By construction, Φ_u is surjective, since $\Phi_u(\text{Vect}(G_{/E(u)})) = E(u)$.
- Φ_u is injective: let $A \in \text{Ker}(\Phi_u)$, so $Au = 0$. Let $x \in E(u)$, then by above there exists $B \in \text{Vect}(G_{/E(u)})$ such that $x = Bu$. As $A \in \text{Ker}(\Phi_u) \subset \text{vect}(G_{/E(u)})$ then $AB = BA$. Therefore $Ax = ABu = BAu = B(0) = 0$. It follows that $A = 0$ and hence $\text{Ker}(\Phi_u) = \{0\}$. \square

Corollary 4.2. *We have $\Phi_u^{-1}(G(u)) = G_{/E(u)}$ and $\Phi_u^{-1}(g_u) = g_{/E(u)}$.*

Under the above notation, we have:

Proposition 4.3. *Let G be an abelian subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$. Then*

$$\exp(\Phi_u^{-1}(E(u))) \subset \Phi_u^{-1}(U_u).$$

To prove Proposition 4.3, we need the following Lemma:

Lemma 4.4. *Let G be an abelian subgroup of $\mathbb{T}_n^*(\mathbb{C})$. If $E(u) = \mathbb{C}^n$, then*

$$\exp(\Phi_u^{-1}(\mathbb{C}^n)) \subset \Phi_u^{-1}(\mathbb{C}^* \times \mathbb{C}^{n-1}).$$

Proof. Here $G_{/E(u)} = G$ and $U_u = U = \mathbb{C}^* \times \mathbb{C}^{n-1}$. First, one has $\exp(\mathcal{G}) \subset \mathcal{G}$: indeed; for every $A \in \mathcal{G}$ we have $A^k \in \mathcal{G}$, for every $k \in \mathbb{N}$ and so $e^A = \sum_{k \in \mathbb{N}} \frac{A^k}{k!} \in \mathcal{G}$. Moreover, we can check that \mathcal{G} is the subalgebra of $M_n(\mathbb{C})$ generated by G .

Second, by corollary 4.2, $\Phi_u^{-1}(G(u)) = G$. Since $E(u) = \mathbb{C}^n$ and by Proposition 4.1, Φ_u is an isomorphism, then

$$\begin{aligned} \exp(\Phi_u^{-1}(\mathbb{C}^n)) &= \exp(\Phi_u^{-1}(E(u))) \\ &= \exp(\mathcal{G}) \\ &\subset \mathcal{G} = \Phi_u^{-1}(\mathbb{C}^n) \end{aligned}$$

Therefore

$$\exp(\Phi_u^{-1}(\mathbb{C}^n)) \subset \Phi_u^{-1}(\mathbb{C}^n) \quad (1)$$

On the other hand, by construction of Φ_u and U , we have $u \in U$ and so $\Phi_u^{-1}(\mathbb{C}^n) \cap \mathbb{T}_n^*(\mathbb{C}) = \Phi_u^{-1}(\mathbb{C}^* \times \mathbb{C}^{n-1})$. As $\exp(\Phi_u^{-1}(\mathbb{C}^n)) \subset \mathbb{T}_n^*(\mathbb{C})$ then by (1) we obtain

$$\exp(\Phi_u^{-1}(\mathbb{C}^n)) \subset \Phi_u^{-1}(\mathbb{C}^n) \cap \mathbb{T}_n^*(\mathbb{C}) = \Phi_u^{-1}(\mathbb{C}^* \times \mathbb{C}^{n-1}).$$

□

Proof of Proposition 4.3. Write $u = [u_1, \dots, u_r]^T \in \mathbb{C}^n$. Suppose that $E(u) = \mathbb{C}^n$, otherwise we replace G by $G_{/E(u)}$. So $G_{/E(u)} = G$ and $U_u = U$. Since $\exp/\mathcal{K}_{\eta,r}(\mathbb{C}) = \exp/\mathbb{T}_{n_1}(\mathbb{C}) \oplus \dots \oplus \exp/\mathbb{T}_{n_r}(\mathbb{C})$ and $\Phi_u^{-1} = \Phi_{u_1}^{-1} \oplus \dots \oplus \Phi_{u_r}^{-1}$. Then by Lemma 4.4

$$\exp(\Phi_u^{-1}(U)) = \prod_{k=1}^r \exp/\mathbb{T}_{n_k}(\mathbb{C}) (\Phi_{u_k}^{-1}(\mathbb{C}^{n_k})) \subset \prod_{k=1}^r \Phi_{u_k}^{-1}(\mathbb{C}^* \times \mathbb{C}^{n_k-1}) = \Phi^{-1}(U).$$

□

As consequence of Proposition 4.3, we have the following results:

Corollary 4.5. *The map $f = \Phi_u \circ \exp/\Phi(\mathbb{C}^n) \circ \Phi_u^{-1} : E(u) \longrightarrow U_u$ is well defined and continuous.*

5. Proof of Theorems 1.2, 1.5, 1.6 and Corollaries 1.3 and 1.4

By Proposition 2.8, suppose that G is an abelian subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$ and then $\mathfrak{g} = \exp^{-1}(G) \cap \mathcal{K}_{\eta,r}(\mathbb{C})$.

In all this section fixed $u \in \mathbb{C}^n$ and suppose that $E(u) = \mathbb{C}^n$ and $G_{/E(u)} = G$, leaving to replace G by $G_{/E(u)}$. Denote by $\Phi = \Phi_u^{-1} : \mathbb{C}^n \longrightarrow \mathcal{G}$. By Corollary 2.2, $\exp : \mathcal{K}_{\eta,r}(\mathbb{C}) \longrightarrow \mathcal{K}_{\eta,r}^*(\mathbb{C})$ is a local diffeomorphism and by Proposition 4.1, Φ is an open map. Then we introduce the following Lemma which will be used in the proof of Theorem 1.2.

Lemma 5.1. *Let O' be an open subset of \mathbb{C}^n such that $\exp/\Phi(O') : \Phi(O') \longrightarrow \exp(\Phi(O'))$ is a diffeomorphism. Then*

$$\exp(\Phi(O')) \cap \overline{\exp(\mathfrak{g})} = \exp(\Phi(O') \cap \overline{\mathfrak{g}}).$$

Proof. Let $A \in \exp(\Phi(O')) \cap \overline{\exp(\mathfrak{g})}$, then $A = e^{\Phi(x)}$ for some $x \in O'$. By Lemma 2.7.(ii), $\exp(\mathfrak{g}) = G$, so $A \in G$. Since $A \in \overline{\exp(\mathfrak{g})}$, there exists a sequence $(B_m)_{m \in \mathbb{N}}$ in \mathfrak{g} such that $\lim_{m \rightarrow +\infty} e^{B_m} = A = e^{\Phi(x)}$. Since $\exp_{/\Phi(O')} : \Phi(O') \rightarrow \exp(\Phi(O'))$ is a diffeomorphism, $\exp(\Phi(O'))$ is an open set containing e^A , so $e^{B_m} \in \exp(\Phi(O'))$, $\forall m \geq p$, for some $p \in \mathbb{N}$. Then $B'_m = \exp_{/\Phi(O')}^{-1}(e^{B_m}) \in \Phi(O')$, $\forall m \geq p$. Since $e^{B_m} \in \exp(\mathfrak{g}) = G$ and $\Phi(O') \subset \mathcal{K}_{\eta,r}(\mathbb{C})$, then $B'_m \in \exp^{-1}(G) \cap \mathcal{K}_{\eta,r}(\mathbb{C}) = \mathfrak{g}$, for every $m \geq p$. Therefore $\lim_{m \rightarrow +\infty} B'_m = \Phi(x)$, so $\Phi(x) \in \Phi(O') \cap \overline{\mathfrak{g}}$ and hence $A \in \exp(\Phi(O')) \cap \overline{\mathfrak{g}}$.

Conversely, by continuity of $\exp_{/\Phi(O')} : \Phi(O') \rightarrow \exp(\Phi(O'))$, one has

$$\exp(\Phi(O')) \cap \overline{\mathfrak{g}} \subset \exp(\Phi(O')) \cap \exp(\overline{\mathfrak{g}}) \subset \exp(\Phi(O')) \cap \overline{\exp(\mathfrak{g})}.$$

The prove is completed. \square

5.1. Proof of Theorem 1.5. The equivalence (i) \iff (ii) follows from Theorem 1.1.(i).

(i) \iff (iii): By Lemma 3.1, suppose that $E(u) = \mathbb{C}^n$ and $U_u = U$ (leaving to replace G by $G_{/E(u)}$). Then by Proposition 4.1 and Corollary 4.2, $\Phi = \Phi_u^{-1} : \mathbb{C}^n \rightarrow \mathcal{G}$ is an isomorphism satisfying $\Phi(G(u)) = G$ and $\Phi(\mathfrak{g}_u) = \mathfrak{g}$.

By Proposition 2.6.(i), there exist a vector space V , contained in $\overline{\mathfrak{g}_u}$, a vector space W such that $V \oplus W = \mathbb{C}^n$ and $\overline{\mathfrak{g}_u} = (\overline{\mathfrak{g}_u} \cap W) \oplus V$, with $\overline{\mathfrak{g}_u} \cap W$ is discrete.

By corollary 2.5 and Proposition 2.6, there exists an open subset O of \mathbb{C}^n such that $O \cap \overline{\mathfrak{g}_u} = V$. By Corollary 2.2, the exponential map $\exp : \mathcal{K}_{\eta,r}(\mathbb{C}) \rightarrow \mathcal{K}_{\eta,r}^*(\mathbb{C})$ is a locally diffeomorphism, then there exists an open subset $O' \subset O$, of \mathbb{C}^n such that the restriction $\exp_{/\Phi(O')} : \Phi(O') \rightarrow \exp(\Phi(O'))$ of the exponential map on $\Phi(O')$ is a diffeomorphism. Since $O' \subset O$, then

$$O' \cap \overline{\mathfrak{g}_u} = O' \cap V. \quad (1)$$

Since $O' \subset U_u$ then by Corollary 4.5 the map $f_{/O'} = \Phi^{-1} \circ \exp_{/\Phi(O')} \circ \Phi : O' \rightarrow O'$ is well defined and as $\exp_{/\Phi(O')}$ is a diffeomorphism then $f_{/O'}$ is a diffeomorphism. By Lemma 2.7, one has $\exp(\mathfrak{g}) = G$ and then:

$$\begin{aligned} f(O') \cap \overline{G(u)} &= \Phi^{-1} \circ \exp(\Phi(O')) \cap \overline{G(u)} \\ &= \Phi^{-1} \left(\exp(\Phi(O')) \cap \overline{\Phi(G(u))} \right) \\ &= \Phi^{-1} \left(\exp(\Phi(O')) \cap \overline{G} \right) \\ &= \Phi^{-1} \left(\exp(\Phi(O')) \cap \overline{\exp(\mathfrak{g})} \right) \end{aligned}$$

By Lemma 5.1 and by (1), we obtain

$$\begin{aligned}
f(O') \cap \overline{G(u)} &= \Phi^{-1} \circ \exp(\Phi(O') \cap \overline{g}) \\
&= \Phi^{-1} \circ \exp(\Phi(O') \cap \Phi(\overline{g_u})) \\
&= \Phi^{-1} \circ \exp \circ \Phi(O' \cap \overline{g_u}) \\
&= \Phi^{-1} \circ \exp \circ \Phi(O' \cap V) \\
&= f(O' \cap V).
\end{aligned}$$

As $f(O')$ is an open subset of \mathbb{C}^n and $O' \cap V$ is an open subset of the real vector space V , then $f(O') \cap \overline{G(u)}$ is a manifold with dimension $m = \dim(V) = \dim(\overline{g_u})$. We conclude that $G(u)$ is regular with order m . Since \mathbb{C}^n is a real vector space with dimension $2n$ and V is a real subspace of \mathbb{C}^n with dimension m , so $m \leq 2n$. We conclude the equivalence (i) \iff (iii). \square

5.2. Proof of Corollary 1.3.

\diamond *Complex Case:* Suppose that $\mathbb{K} = \mathbb{C}$. Let G be an abelian subgroup of $GL(n, \mathbb{C})$ and $u \in \mathbb{C}^n$. By Theorem 1.5, the orbit $G(u)$ is regular with order m if and only if $\dim(\overline{g_u}) = m$. Since $\overline{g_u}$ is a closed additive subgroup of \mathbb{C}^n (Lemma 2.7.(i)). Then $\dim(\overline{g_u}) \geq 0$, so $G(u)$ is regular with order $m \geq 0$. This proves the complex case.

\diamond *Real Case:* Suppose that $\mathbb{K} = \mathbb{R}$. Let G be an abelian subgroup of $GL(n, \mathbb{R})$ and $x \in \mathbb{R}^n$. So G is considered as an abelian subgroup of $GL(n, \mathbb{C})$. By the above case, $G(x)$ is regular with some order m , with $m \leq 2n$. Then there exists an open subset $O = O_1 + iO_2$ of \mathbb{C}^n with O_1, O_2 are open subsets of \mathbb{R}^n , such that $\overline{G(x)} \cap O$ is a manifold with dimension m . One has $0 \in O_2$, since $G(x) \subset \mathbb{R}^n$. Then $\overline{G(x)} \cap O = \overline{G(x)} \cap O_1$ and $m \leq n$. It follows that $G(x)$ is a regular orbit in \mathbb{R}^n with order m . The proof is completed. \square

5.3. Proof of Corollary 1.4.

Lemma 5.2. ([1] Corollary 1.3). *If G has a locally dense orbit γ in \mathbb{C}^n then γ is dense in \mathbb{C}^n .*

Proof of Corollary 1.4.

(i) If $\overline{G(u)} = \mathbb{R}^n$ then $\overline{G(u)}$ is a manifold with dimension n , so $G(u)$ is regular with order $m = n$.

Conversely, if $G(u)$ is regular with order $m = n$, then $\overline{G(u)} \cap O$ is a manifold with order $m = n$, for some open subset O of \mathbb{R}^n . Hence $\overline{G(u)} \cap O$ is an open subset of \mathbb{R}^n . Therefore $G(u)$ is locally dense.

(ii) We use the same proof of (i) and by Lemma 5.2 we have $\overline{G(u)} = \mathbb{C}^n$. \square

5.4. Proof of theorem 1.6. If $\overline{G(u)}$ is a vector subspace then $\overline{G(u)} = E(u)$, so $G(u)$ is regular with order $2r(u)$.

Conversely, if $G(u)$ is regular with order $2r(u)$, then $\overline{G(u)} \cap O$ is a manifold with dimension $2r(u)$, for some open set O . Since $\dim(E(u)) = 2r(u)$ over \mathbb{R} , then $\overline{G(u)} \cap O$ is an open subset of $E(u)$. So $G(u)$ is locally dense in $E(u)$. By lemma 5.2 applied on $G/E(u)$ we have $\overline{G(u)} = E(u)$. \square

5.5. Algebraic Lemmas.

Lemma 5.3. ([11], Proposition 4.3). Let $H = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_p$ with $u_k = [u_{k,1}, \dots, u_{k,n}]^T \in \mathbb{R}^n$, $k = 1, \dots, p$. Then H is dense in \mathbb{R}^n if and only if for every $(s_1, \dots, s_p) \in \mathbb{Z}^p - \{0\}$:

$$\text{rank} \left(\begin{bmatrix} u_{1,1} & \dots & \dots & u_{p,1} \\ \vdots & \vdots & \vdots & \vdots \\ u_{1,n} & \dots & \dots & u_{p,n} \\ s_1 & \dots & \dots & s_p \end{bmatrix} \right) = n + 1.$$

Corollary 5.4. Let $p \geq n+1$ and $H = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_p$, $u_k \in \mathbb{R}^n$, $1 \leq k \leq p$, such that (u_1, \dots, u_n) is a basis of \mathbb{R}^n . If there exists $0 \leq m \leq n$ such that $u_k = \sum_{j=1}^m \alpha_{k,j} u_{j+n-m}$, for every $n+1 \leq k \leq p$. Then the following assertions are equivalent:

- (i) $\dim(\overline{H}) = m$.
- (ii) u_1, \dots, u_p satisfies property $\mathcal{D}(m)$.

Proof. Let $E = \mathbb{R}u_{n-m+1} \oplus \dots \oplus \mathbb{R}u_n$. We replace \mathbb{R}^n by E in Lemma 5.3 and we obtain: u_1, \dots, u_p satisfies property $\mathcal{D}(m)$ if and only if $K = \mathbb{Z}u_{n-m+1} + \dots + \mathbb{Z}u_p$ is dense in E and this is equivalent to $\dim(\overline{H}) = m$ since $\overline{H} = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_{n-m} + \overline{K}$. \square

5.6. Proof of Theorem 1.6.

Denote by ‘:

- $u_0 = [e_{1,1}, \dots, e_{r,1}]^T$ and $e_{k,1} = [1, 0, \dots, 0]^T \in \mathbb{C}^{n_k}$, $k = 1, \dots, r$.
- $v_0 = Pu_0$, where $P \in GL(n, \mathbb{C})$ is defined in Proposition 4.1 so that $P^{-1}GP \subset \mathcal{K}_{\eta,r}^*(\mathbb{C})$.

Proposition 5.5. ([2], Theorem 1.3) Let G be an abelian subgroup of $GL(n, \mathbb{C})$ generated by A_1, \dots, A_p . Let $B_1, \dots, B_p \in \mathfrak{g}$ such that $A_k = e^{B_k}$, $k = 1, \dots, p$. Then $\mathfrak{g}_{v_0} = \sum_{k=1}^p \mathbb{Z}B_k v_0 + \sum_{k=1}^r 2i\pi \mathbb{Z}Pe^{(k)}$.

By construction of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$, remark that for every $u \in U$ there exists $Q \in \mathcal{K}_{\eta,r}^*(\mathbb{C})$ such that $Qu_0 = u$. Then as a consequence of Proposition 5.5 we get the following Corollary:

Corollary 5.6. *Let G be an abelian subgroup of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$ generated by A_1, \dots, A_p . Let $B_1, \dots, B_p \in \mathfrak{g}$ such that $A_k = e^{B_k}$, $k = 1, \dots, p$. Then for every $u \in U$, $g_u = \sum_{k=1}^p \mathbb{Z}B_k u + \sum_{k=1}^r 2i\pi \mathbb{Z}Qe^{(k)}$, where $Q \in \mathcal{K}_{\eta,r}^*(\mathbb{C})$ such that $Qu_0 = u$.*

Proof of Theorem 1.6. The proof of Theorem 1.6 results from Theorem 1.2, Corollary 5.4 and Corollary 5.6. \square

6. Examples

Let $\mathbb{D}_n(\mathbb{C}) = \{A = \text{diag}(a_1, \dots, a_n) : a_k \in \mathbb{C}^*, 1 \leq k \leq n\}$ and let G be an abelian subgroup of $\mathbb{D}_n(\mathbb{C})$. In this case we have $\tilde{G} = G$ is a subgroup of $\mathcal{K}_{(1,\dots,1),n}(\mathbb{C})$ and $r_G = n$.

Example 6.1. Let G be the group generated by $A_k = \text{diag}(\lambda_{k,1}e^{i\alpha_{k,1}}, \dots, \lambda_{k,n}e^{i\alpha_{k,n}})$, $k = 1, \dots, p$, where $\lambda_{k,j} \in \mathbb{R}_+$, $\alpha_{k,j} \in \mathbb{R}$, $1 \leq j \leq n$. Let $u = [x_1, \dots, x_n]^T \in \mathbb{C}^n$, then the following assertions are equivalent:

- (i) $G(u)$ is regular with order m .
- (ii) $u_k = [(log \lambda_{k,1} + i\alpha_{k,1})x_1, \dots, (log \lambda_{k,n} + i\alpha_{k,n})x_n]^T$, $1 \leq k \leq p$ with $2i\pi e_1, \dots, 2i\pi e_n$, satisfies $\mathcal{D}(m)$.

Proof. We let $B_k = \text{diag}(log \lambda_{k,1} + i\alpha_{k,1}, \dots, log \lambda_{k,n} + i\alpha_{k,n})$,

One has $e^{B_k} = A_k$ and $B_k \in \mathbb{D}_n(\mathbb{C})$, $1 \leq k \leq p$. Then $B_k \in \mathfrak{g}$. The result follows then from Theorem 1.6. \square

Example 6.2. ([1], Example 6.2) Let G be the subgroup of $GL(4, \mathbb{R})$ generated by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then:

- i) if $u \in (\mathbb{Q}^*)^2 \times \mathbb{R}^2$, $G(u)$ is discrete. So $G(u)$ is regular with order 0.
- ii) if $u \in \mathbb{Q}^* \times (\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R}^2$, $G(u)$ is dense in a straight line. So $G(u)$ is regular with order 1.

In this example, G is considered as a subgroup of $GL(4, \mathbb{C})$. We have $\tilde{G} = G$ is a subgroup of $\mathbb{T}_4^*(\mathbb{C})$ and $G(e_1) = \{[1, 0, 0, n+m]^T, \quad n, m \in \mathbb{Z}\}$, then $E(e_1) = \mathbb{C}e_1 + \mathbb{C}e_4$ and $U_{e_1} = \mathbb{C}^*e_1 + \mathbb{C}e_4$. So $E(e_1) \neq \mathbb{C}^n$. Let $B_1 = A_1 - I_4$ and $B_2 = A_2 - I_4$. Since $B_1^2 = B_2^2 = 0$, so $e^{B_1} = A_1$ and $e^{B_2} = A_2$. By Theorem 1.6, $g_u = \mathbb{Z}B_1u + \mathbb{Z}B_2u + 2i\pi\mathbb{Z}e_1$. For every $u = [x, y, z, t]^T \in \mathbb{R}^* \times \mathbb{R}^3$, we have $g_u = (\mathbb{Z}x + \mathbb{Z}y)e_4 + 2i\pi\mathbb{Z}e_1$, then:

-if $\frac{y}{x} \notin \mathbb{Q}$, then $\overline{g_u} = \mathbb{R}e_4 + \mathbb{Z}e_1$, so $\dim(\overline{g_u}) = 1$. By Theorem 1.6, $G(u)$ is regular with order 1.
 -if $\frac{y}{x} \in \mathbb{Q}$, then $\overline{g_u} = \mathbb{Z}ae_4 + \mathbb{Z}e_1$, for some $a \in \mathbb{R}$, so $\dim(\overline{g_u}) = 0$. By Theorem 1.6, $G(u)$ is regular with order 0. \square

A simple example for $n = 1$ and $\mathbb{K} = \mathbb{R}$ is given in the following, to show that the closure of a regular orbit is not necessarily a manifold.

Example 6.3. Let $\lambda > 1$ and G be the group generated by $\lambda \cdot id_{\mathbb{R}}$, then for every $x \in \mathbb{R}^*$, we have $G(x) = \{\lambda^n, \quad n \in \mathbb{Z}\}$ and $\overline{G(x)} = \{\lambda^n, \quad n \in \mathbb{Z}\} \cup \{0\}$. Thus $G(x)$ is discrete, so it is regular with order 0, but $\overline{G(x)}$ is not a manifold.

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